# Key Principles and Processes for Teaching Mathematics for Deep Understanding

**Principles of Teaching and Learning Mathematics**

The goal of the Core Curriculum is to provide an opportunity for **every** student to learn significant mathematics with a deep understanding of the concepts taught. A major premise is that **all students can learn mathematics and need to be held to high expectations**. To achieve this goal, the curriculum must be more than a collection of activities combined with sets of memorized facts and rules. The curriculum must be challenging and focused on important mathematics concepts that are well articulated across grades and subjects.

Teachers need to:

- Understand what students need to know and be able to do.
- Be well prepared with an extensive knowledge of mathematics and pedagogically sound instructional techniques.
- Adapt classroom instruction to meet the diverse learning styles of their students.

Students need to:

- Receive strong support from their teachers, parents, and an informed society that recognizes the importance of a comprehensive mathematics education.
- Participate in activities that build and strengthen a deep and profound understanding of why and how we do mathematics.
- Apply the mathematical ideas they are learning and be able to make connections to their world and prospective occupations.
- Connect their prior knowledge and experience to new ideas.

Assessment must be:

- An integral part of the curriculum and a routine part of classroom instruction.
- Rich and varied and used continuously to make sure that a high-quality, effective learning experience is available for every student.
- One of the primary sources for making instructional decisions regarding how, when, and what mathematical concepts are to be taught.

Technology:

- Must be integrated into the curriculum and used appropriately as part of mathematical instruction.
- Provides visual images when teaching mathematical ideas and concepts.
- Facilitates the organization and analysis of data.
- Facilitates efficiency and accuracy in computation.
- Enhances the investigation and modeling of a wide variety of mathematical concepts.

**Importance of Mathematics**

If Utah students are to function effectively in a time of extraordinary and accelerating change, they must understand and be able to use mathematics in both their personal and professional lives. There has never been a greater need to be mathematically literate than in our rapidly expanding society and economy. Those who understand and can do mathematics will have significantly enhanced opportunities and options that will open doors to productive futures. Those who lack mathematical competence will find many doors leading to productive and successful futures closed.

Society has many misconceptions about mathematics and its role in our world.

- Mathematics is incorrectly viewed as a collection of rigid rules and mysterious procedures that seem to be unrelated to each other and require total mastery with little or no understanding.
- Mathematics is perceived by many to be difficult and demanding and is considered to be a subject in which it is socially acceptable to do poorly.
- The pervasive role of mathematics is underestimated in both the world of work and the world of everyday living.
- Mathematics learned in school is considered to be irrelevant, unnecessary, and unrelated to the mathematics students will encounter in their professional and personal lives.

These false perceptions and regrettable attitudes about mathematics have a significant and negative impact on mathematics education. These attitudes about mathematics can be changed only as our students become knowledgeable about mathematics and are prepared to become lifelong learners and users of mathematics.

## Processes of Teaching and Learning Mathematics

Students need to know and be able to use basic mathematical facts and procedures. However, current research* makes clear that **how** mathematics is taught is as important or more important than the mathematical concepts being taught. A deep conceptual understanding is the foundation upon which mathematical proficiency, factual knowledge, and procedural skills are built. In *Principles and Standards of School Mathematics,* published by the National Council of Teachers of Mathematics, five Process Standards are identified that highlight ways of acquiring and using mathematical content knowledge. These Process Standards are Problem Solving, Reasoning and Proof, Communication, Connections, and Representation.

### All secondary students in Utah will engage in the following types of activities in their study of mathematics.

**Problem Solving**

1. Select and use appropriate methods for computing, e.g., mental computation, estimation, paper and pencil, and calculator or computer.
2. Propose, critique, and value alternative approaches to solving problems.
3. Extend mathematical knowledge by considering the thinking strategies of others.
4. Reflect and evaluate mathematical thinking processes used in solving problems.
5. Utilize different problem solving strategies including:
    a. Drawing a picture or diagram.
    b. Looking for a pattern.
    c. Identifying counterexamples.
    d. Choosing an appropriate operation.
    e. Guessing and checking.

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* *Principles and Standards of School Mathematics,* National Council of Teachers of Mathematics, Inc., Virginia, 2000, pages 383-390.

    f. Making a list, table, graph, or equation.
    g. Working backwards.
    h. Eliminating possibilities.
    i. Making a model or simulation.
    j. Solving a simpler or related problem.
    k. Checking the reasonableness of results.
    l. Using proportional reasoning.
6. Develop clarification and understanding of new mathematical concepts, processes, and vocabulary by reflecting upon and answering such questions as:
    a. What made you think of that?
    b. Did anyone think about this in a different way?
    c. How are these ideas related?
    d. Where have we seen a problem like this before?
    e. How does today's work relate to what we have done in earlier units of study?
7. Solve a variety of multi-step, non-routine, complex problems including puzzles, applications, patterning, and open-ended or extended problem-solving projects.
8. Estimate solutions to problems and determine the reasonableness of answers by relating them to the estimates.

**Reasoning and Proof**

1. Link problem solving to the sequence of steps in a proof and draw reasonable conclusions.
2. Explain and justify problem-solving procedures.
3. Examine patterns and note regularities and irregularities in various types of problems.
4. Make and investigate mathematical conjectures.
5. Formulate counterexamples.
6. Use a variety of formal and informal proofs appropriate to the course.
7. Identify information as necessary, sufficient, or extraneous and conclusions as valid or invalid.
8. Realize that observing a pattern and stating a conjecture related to the pattern does not constitute a proof.

**Communication**

1. Express mathematical ideas coherently and clearly to peers, teachers, and others.
2. Employ the precise language and notation of mathematics to clearly express mathematical ideas.
3. Organize and consolidate mathematical thinking using communication methods, e.g., class and group discussion, journals, portfolios, oral presentations, and written reports.

**Connections**

1. Formulate real-world situations that require extended investigations, solve them, and justify answers.
2. Establish connections among mathematical expressions, physical models, pictorial representations, and real-world situations.
3. Find applications of mathematical concepts in newspapers, magazines, television, radio, or other sources.
4. Explore historical and multicultural contributions to mathematics.
5. Recognize and apply mathematical ideas and relationships in areas outside the mathematics classroom, e.g., art, science, other curricular areas, and everyday life.

**Representation**

1. Use a variety of visual representations (e.g., patty paper, dot paper, graph paper, models, manipulatives, nets, and technology) to explore and formulate conjectures related to mathematical concepts being studied.

2. Represent mathematical concepts using physical models, visualizations, and appropriate symbolic notations.
3. Represent problem situations verbally, numerically, graphically, geometrically, or algebraically.

# Geometry

## Prerequisite: Mastery of Elementary Algebra or Applied Mathematics I

## Course Description

Students in Geometry study Euclid's postulates and theorems as the basis for an axiomatic system. Students will explore geometry through inductive and deductive processes, technology, constructions, manipulatives, and algebraic connections. Topics of investigation include logic, angle and line relationships, triangles and other polygons, congruence, and similarity. Students also study coordinate geometry. Trigonometric ratios of sine, cosine, and tangent are used to solve triangle problems. Students will use area, volume, geometric probability, and geometric relationships to solve real-life problems. While mathematical skills will be developed, teaching will focus on the understanding of concepts in depth, enabling students to apply mathematical skills and make meaningful connections to life's experiences.

# Geometry

## Standard 1: Students will acquire number sense and perform operations with real numbers.

There were no new extensions of the number system or number operations introduced in Geometry.

## Standard 2: Students will represent and analyze mathematical situations and properties using patterns, relations, functions, and algebraic symbols.

**Objective 2.1: Use patterns, relations, and functions to represent mathematical situations.**

1. Identify **trigonometric relationships** (**sine, cosine,** and **tangent**) using right triangles, expressing the relationships as fractions or decimals.
2. Analyze geometric patterns to develop formulas and communicate how the formulas were derived, e.g., angle measure and number of sides of a polygon, interior and exterior angles, diagonals, and vertices.
3. Solve problems using the properties of special right triangles, e.g., 30°, 60°, 90° or 45°, 45°, 90°.
4. Identify the effect on area or volume when changing linear dimensions.

**Objective 2.2: Evaluate, solve, and analyze mathematical situations using algebraic properties and symbols.**

1. Find the angle measure in degrees given the trigonometric ratio using a calculator.
2. Find the trigonometric ratio given the angle measure in degrees using a calculator.
3. Find the missing measures of right triangles.
4. Find missing parts of triangles using the **Law of Sines** and **Law of Cosines**.
5. Write an equation of a line perpendicular or parallel to a line through a given point.
6. Model and solve geometric situations using algebraic properties.

## Standard 3: Students will solve problems using spatial and logical reasoning, applications of geometric principles, and modeling.

**Objective 3.1: Analyze characteristics and properties of two- and three-dimensional shapes and develop mathematical arguments about geometric relationships.**

1. Use accepted geometric notations, e.g., congruencies, **transformations**, similarities.
2. Write **conditional statements, converses**, and **inverses** and determine the **truth value** of the statements.
3. Prove a statement false by using a **counterexample**.
4. Identify angle pairs as **adjacent**, **complementary, supplementary, a linear pair,** or **vertical angles**.
5. Differentiate between parallel, perpendicular**, skew,** and intersecting lines.
6. Classify angle pairs formed by two lines and a **transversal**, e.g., corresponding, alternate interior, and supplementary angles.
7. Prove lines parallel or perpendicular using slope or angle relationships.
8. Prove congruency and similarity of geometric figures.
9. Identify **medians**, **altitudes**, and **angle bisectors** of a triangle, and the **perpendicular bisectors** of the sides of a triangle.
10. Classify a **quadrilateral** as a **parallelogram**, **trapezoid**, **rectangle**, **square**, **rhombus**, **kite,** or none of the above.

**Objective 3.2: Specify locations and describe spatial relationships using coordinate geometry.**

1. Graph a circle given the equation in the form $(x - h)^2 + (y - k)^2 = r^2$.
2. Write the equation of a circle given its graph.
3. Verify the classifications of geometric figures using coordinate geometry to find lengths and slopes, e.g., verify or prove the diagonals of a rectangle are congruent using the distance formula.
4. Perform and analyze **transformations** (**translations**, **rotations**, **reflections**, and **dilations**) using coordinate geometry.

**Objective 3.3: Use visualization, spatial reasoning, and geometric modeling to solve problems.**

1. Construct/copy angles and segments, bisect angles and segments, and create perpendicular lines and parallel lines using a compass and straight edge, technology, or other manipulatives.
2. Define p as the ratio of the circumference to the diameter of a circle.
3. Identify the relationships between the measures of **intercepted arcs** and **inscribed** or **central angles**.
4. Solve real-world problems using trigonometric ratios and properties of congruent and similar figures, e.g., "How much paint is needed to paint a room?" or "How can we ensure square corners in a building during construction?"
5. Sketch cross-sections of geometric solids.

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| 11. Identify **radii**, **diameters**, **chords**, **secants**, **arcs**, **sectors**, **central angles**, **inscribed angles**, and **tangents** for circles.<br>12. Classify and use the properties of acute, right, scalene, oblique, isosceles, equilateral, or equiangular triangles.<br>13. Classify polyhedrons and other three-dimensional figures according to their properties. | | |
| **Standard 4: Students will understand and apply measurement tools, formulas, and techniques.** | | |
| | **Objective 4.2: Determine measurements using appropriate techniques, tools, and formulas.**<br>1. Find the area of a regular polygon.<br>2. Find the length of an arc and the area of a **sector**.<br>3. Find the surface area and volume for **prisms**, **cylinders**, **pyramids**, cones, and **spheres** given the formula.<br>4. Estimate the area of an irregular region. | |
| **Standard 5: Students will draw conclusions using concepts of probability after collecting, organizing, and analyzing a data set.** | | |
| | **Objective 5.2: Apply basic concepts of probability**.<br>1. Identify geometric probabilities by performing simulations involving length or area.<br>2. Calculate geometric probability. | |

# Glossary for Geometry

**Adjacent angle** – Any two non-overlapping angles with a side in common.

**Altitude of a triangle** – A segment drawn from a vertex and perpendicular to the opposite side or top the line containing the opposite side.

**Angle bisector** – A ray in the interior of an angle that divides the angle into two congruent angles.

**Arc** – A continuous part of a circle. The measure of an arc is the measure of the angle formed by two radii with endpoints at the endpoints of the arc.

**Central angle** – An angle whose vertex is located at the center of a circle and whose sides are radii of the circle.

**Chord** – A segment whose endpoints are on a circle.

**Complementary angles** – Two angles with measures whose sum is 90°.

**Conditional statement** – A logical statement consisting of two parts, an hypothesis and a conclusion.

**Cone** – A solid bounded by a circular base and a curved surface with one vertex, e.g., an ice cream cone.

**Converse** – In the converse of a conditional statement, the hypothesis and conclusion are reversed.

**Cosine** – In a right triangle, the ratio of the length of the leg adjacent to the reference angle to the length of the hypotenuse.

**Counterexample** – An example that shows that a conjecture is not always true.

**Cylinder** – A solid bounded by two congruent and parallel circular regions joined by a curved surface whose cross-section perpendicular to the axis is always a circle congruent to the bases.

**Diameter** – A chord that contains the center of the circle.

**Dilation** – A transformation in which a similar image is formed by enlarging or reducing its preimage. The image and preimage are similar figures.

**Inscribed angle** – An angle whose vertex is on a circle and whose sides are chords of a circle.

**Intercepted arc** – The portion of a circle that is in the interior of an inscribed angle and whose endpoints are on the sides of the angle.

**Inverse of a conditional statement** – In the inverse of a conditional statement, both the hypothesis and the conclusion are negated.

**Law of Cosines** – In any triangle *ABC* with sides designated *a*, *b*, and *c*, and angles designated *A*, *B*, and *C*, $a^2 = b^2 + c^2 - 2bc \cos A$.

**Law of Sines** – In any triangle *ABC* with sides designated *a*, *b*, and *c,* and angles designated *A*, *B*, and *C*, $\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}$.

**Linear pair of angles** – Two adjacent angles whose noncommon sides are opposite rays.

**Median of a triangle** – A segment from a vertex to the midpoint of the opposite side.

**Perpendicular bisector** – A line or line segment that is perpendicular to a segment at its midpoint.

**Preimage** – The original figure used in a transformation.

**Prism** – A polyhedron that has two congruent and parallel faces called bases joined by faces that are parallelograms.

**Pyramid** – A polyhedron with three or more triangular faces that meet at a point and one other face, a polygon, that is called the base.

**Radius (plural radii)** – A segment or distance from the center of a circle to a point on the circle.

**Reflection** – A transformation in which a figure is flipped over a line called the line of reflection. All corresponding points in the image and preimage are equidistant from the line of reflection.

**Rotation** – A transformation in which the image is formed by turning its preimage about a point.

**Secant of a circle** – A line that intersects a circle in two points.

**Sector of a circle** – The region bounded by two radii of a circle and the arc they intercept.

**Sine** – In a right triangle, the ratio of the length of the leg opposite the reference angle to the length of the hypotenuse.

**Skew lines** – Non-coplanar lines that do not intersect.

**Sphere** – All points in space equidistant from a given point.

**Supplementary angles** – Two angles with measures whose sum is 180°.

**Tangent** – In a right triangle, the ratio of the length of a leg opposite the reference angle to the length of the leg adjacent to the given angle.

**Tangent to a circle** – A segment drawn from a point external to a circle, tangent to a circle at its other endpoint.

**Transformation** – An operation that creates an image from an original figure or preimage.

**Translation** – A transformation in which an image is formed by moving every point of a figure the same distance in the same direction.

**Transversal** – A line that intersects two or more other lines at different points.

**Trigonometric relationship** – A ratio that compares the length of two sides of a right triangle.

**Truth value** – The truth or falseness of a statement.

**Vertical angles** – The non-adjacent angles formed by intersecting lines.